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Positive Solutions of Singular Differential Equations on Measure Chains

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Abstract— This paper is concerned with singular differential equations on measure chains. Theorems on the existence of positive solutions and eigenvalue intervals are obtained, which extend some existing results.

Keywords— Singular, positive solutions, Measure chains, Eigenvalue, Fixed points.

1. INTRODUCTION AND PRELIMINARIES

The theory of measure chains was introduced and developed by Aulbach and Hilger [1] in 1988 to unify continuous and discrete analysis. Some other early papers on this topic are [2,3]. There has been increasing interest in studying this theory all these years. Recently, much attention is attracted by questions of existence of positive solutions to boundary value problems for differential equations on measure chains. For significant works along this line, see, e.g., [4–18]. Stimulated by these works, we investigate in this paper the existence of positive solutions of boundary value problems for *singular* differential equations on measure chains.

Before discussing the problems of interest for this paper, we recall some definitions and notations which are common to the recent literature. Our sources for this background material are the papers [1–18].

DEFINITION 1.1. *Let $\mathbb{T}$ be a nonempty closed subset of $\mathbb{R}$, the set of real numbers, with the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. Define the forward (resp., backward)*

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jump operator σ *(resp.,* ϖ*) by*

$$\sigma(t) := \inf\{\tau > t; \tau \in \mathbb{T}\} \in \mathbb{T}, \qquad (\text{resp., } \varpi(t) := \sup\{\tau < t;\ \tau \in \mathbb{T}\} \in \mathbb{T},),$$

for all $t \in \mathbb{T}$ *with* $t < \sup \mathbb{T}$ *(resp.,* $t > \inf \mathbb{T}$*). If* $\sigma(t) > t$ *(resp.,* $\varpi(t) < t$*), we say* t *is right (resp., left) scattered. If* $\sigma(t) = t$ *(resp.,* $\varpi(t) = t$*), we say* t *is right (resp., left) dense.*

Throughout this paper, we make the basic assumption that $a < b$ are points in $\mathbb{T}$, such that either $\sigma^2(b) > \sigma(b) > b$ or $\sigma(b) = b$ and b is left dense.

DEFINITION 1.2. *Define the interval in* $\mathbb{T}$

$$[a, b] := \{t \in \mathbb{T}, \text{ such that } a \le t \le b\}.$$

Other types of intervals are defined similarly.

DEFINITION 1.3. *Assume* $x : [a, \sigma^2(b)] \to \mathbb{R}$ *and fix* $t \in [a, \sigma(b)]$*. Then, we define* $x^\Delta(t)$ *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$*, there is a neighborhood* U *of* t*, such that*

$$\left| [x(\sigma(t)) - x(s)] - x^\Delta(t) [\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s|,$$

for all $s \in U \cap [a, \sigma^2(b)]$*. We call* $x^\Delta(t)$ *the delta derivative of* $x(t)$*. The second derivative of* $x(t)$ *(*$t \in [a, b]$*) is defined by* $x^{\Delta\Delta}(t) = (x^\Delta)^\Delta(t)$*.*

It can be shown that if x is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, the set of integers, then

$$x^\Delta(t) = \Delta x(t) := x(t+1) - x(t).$$

In particular, if $\mathbb{T} = \mathbb{R}$, then $x^\Delta(t)$ reduces to the usual derivative $x'(t)$.

DEFINITION 1.4. *If* $F^\Delta(t) = f(t)$*, then we define an integral by*

$$\int_a^t f(\tau) \Delta\tau = F(t) - F(a).$$

For the definitions and basic properties of the more general Riemann integrals and improper integrals on time scales, we refer the reader to [5] and references therein.

In this paper, we are concerned with the existence of positive solutions of the following problem

$$\begin{aligned} [\rho(t) x^\Delta(t)]^\Delta + m(t) f(t, x(\sigma(t))) &= 0, \qquad t \in [a, b], \\ \alpha x(a) - \beta x^\Delta(a) &= 0, \\ \gamma x(\sigma(b)) + \delta x^\Delta(\sigma(b)) &= 0, \end{aligned} \tag{1.1}$$

and its eigenvalue problem

$$\begin{aligned} [\rho(t) x^\Delta(t)]^\Delta + \lambda m(t) f(t, x(\sigma(t))) &= 0, \qquad t \in [a, b], \\ \alpha x(a) - \beta x^\Delta(a) &= 0, \\ \gamma x(\sigma(b)) + \delta x^\Delta(\sigma(b)) &= 0, \end{aligned} \tag{1.1$_\lambda$}$$

where $\rho(t) > 0$ on $[a, \sigma(b)]$, such that both the delta derivative of $\rho(t)$ and the integral $\int_a^{\sigma(b)} (1/\rho(\tau)) \Delta\tau$ exist, $m(\cdot)$ and $f(\cdot, \cdot)$ are given functions, $\alpha, \beta, \gamma, \delta \ge 0$, such that

$$d := \frac{\gamma\beta}{\rho(a)} + \frac{\alpha\delta}{\rho(\sigma(b))} + \alpha\gamma \int_a^{\sigma(b)} \frac{1}{\rho(\tau)} \Delta\tau > 0. \tag{1.2}$$

In 2000, Erbe and Peterson [13] studied the existence of positive solutions of the following differential equation on a measure chain

$$\begin{aligned} -x^{\Delta\Delta}(t) &= f(t, x(\sigma(t))), \qquad t \in [a, b], \\ \alpha x(a) - \beta x^\Delta(a) &= 0, \\ \gamma x(\sigma(b)) + \delta x^\Delta(\sigma(b)) &= 0. \end{aligned} \tag{1.3}$$

They proved the following.

THEOREM 1.5. *(See [13, Theorem 9].) If either the superlinear case* $f_0 = 0, f_\infty = \infty$ *or the sublinear case* $f_0 = \infty, f_\infty = 0$ *holds, then the BVP (1.3) has a positive solution, where* $f \in C([a, \sigma(b)] \times \mathbb{R}^+, \mathbb{R}^+)$ *and* $f_0 := \lim_{x\to 0^+}(f(t,x)/x), f_\infty := \lim_{x\to\infty}(f(t,x)/x)$.

Chyan and Henderson [6] investigated the existence of positive solutions of the following differential equation on a measure chain

$$x^{\Delta\Delta}(t) + \lambda a(t) f(x(\sigma(t))) = 0, \qquad t \in (0,1), \tag{1.4}$$

subject to either the conjugate boundary value condition

$$x(0) = x(\sigma(1)) = 0, \tag{1.5}$$

or the right focal boundary value condition

$$x(0) = 0 = x^{\Delta}(\sigma(1)), \tag{1.6}$$

where

(A) $f \in C([0,\infty), [0,\infty))$;
(B) $a(t) \in C([0,\sigma(1)], [0,\infty))$ does not vanish identically on any subinterval of $[0,\sigma(1)]$;
(C) $f_0 := \lim_{x\to 0^+}(f(x)/x)$ and $f_\infty := \lim_{x\to\infty}(f(x)/x)$ exist and are positive.

The following two theorems were obtained by Chyan and Henderson [6].

THEOREM 1.6. *(See [6, Theorem 3.1].) Assume that Conditions (A), (B), and (C) are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{m f_\infty \int_\xi^\omega G(\tau,s)a(s)\Delta s} < \lambda < \frac{\sigma(1)}{f_0 \int_0^{\sigma(1)} \sigma(s)(\sigma(1)-\sigma(s))a(s)\Delta s},$$

for some suitable $\xi, \omega \in [\sigma(1)/4, 3\sigma(1)/4]$ *defined by* $\xi := \min\{t \in \mathbb{T};\ t > \sigma(1)4\}$ *and* $\omega = \max\{t \in \mathbb{T};\ t \le 3\sigma(1)/4\}$, *respectively, where* $G(t,s)$ *is the Green function of* $x^{\Delta\Delta}(t) = 0$ *with respect to boundary value condition (1.5),*

$$m = \min\{1/4, c\}, \qquad c = \min_{s\in[0,\sigma(1)]} \frac{G(\sigma(\omega), s)}{G(\sigma(s)), s)},$$

$$\int_\xi^\omega G(\tau,s)\Delta s = \max_{t\in[\xi,\omega]} \int_\xi^\omega G(t,s)\Delta s, \tag{1.7}$$

the BVP (1.4) with (1.5) has at least one positive solution.

THEOREM 1.7. *(See [6, Theorem 3.2].) Assume that Conditions (A), (B), and (C) are satisfied. Then, for each* λ *satisfying*

$$\frac{1}{m f_0 \int_\xi^\omega G(\tau,s)a(s)\Delta s} < \lambda < \frac{\sigma(1)}{f_\infty \int_0^{\sigma(1)} \sigma(s)(\sigma(1)-\sigma(s))a(s)\Delta s}, \tag{1.8}$$

where $G(t,s)$ *is the Green function of* $x^{\Delta\Delta}(t) = 0$ *with respect to boundary value condition (1.6), numbers* ξ, ω, *and* m *are the same as those of Theorem 1.6, the BVP (1.4) with (1.6) has at least one positive solution.*

In [9], Davies, Henderson, Prasad and Yin also considered the nonlinear conjugate eigenvalue problem (1.4) and (1.5), but allowed the coefficient function $a(t)$ to have a singularity at the left and/or the right endpoints.

Recently, Hong and Yeb [16] studied the following boundary value problem, which is more general than problems (1.4) with (1.5), and (1.4) with (1.6),

$$\begin{gathered} x^{\Delta\Delta}(t) + \lambda f(t, x(\sigma(t))) = 0, \qquad t \in [0,1], \\ \alpha x(0) - \beta x^{\Delta}(0) = 0, \\ \gamma x(\sigma(1)) + \delta x^{\Delta}(\sigma(1)) = 0, \end{gathered} \tag{1.9}$$

where $f \in C([0,\sigma(1)] \times [0,\infty), [0,\infty))$. They showed the following.

THEOREM 1.8. *(See [16, Theorem 3.1].) Assume that Conditions (C_1)-(C_5) hold. If*

$$I_f := \left(\frac{1}{M(\min f_\infty) \int_\xi^\omega G(\tau, s)\Delta s}, \frac{1}{(\max f_0) \int_0^{\sigma(1)} G(\sigma(s), s)\Delta s} \right)$$

is nonempty, then there exists at least one positive solution of the BVP (1.9) for each $\lambda \in I_f$, where (C_1)-(C_5) are stated in [16, Section 2, p. 501–502].

THEOREM 1.9. *(See [16, Theorem 3.2].) Assume that Conditions (C_1)–(C_5) hold. If*

$$I_f := \left(\frac{1}{M(\min f_0) \int_\xi^\omega G(\tau, s)\Delta s}, \frac{1}{(\max f_\infty) \int_0^{\sigma(1)} G(\sigma(s), s)\Delta s} \right)$$

is nonempty, then there exists at least one positive solution of the BVP (1.9) *for each $\lambda \in I_f$, where (C_1)–(C_5) are stated in [16, Section 2, p. 501–502].*

We also note that, Henderson and Wang [15] studied the eigenvalue problem

$$x''(t) + \lambda f(t, x) = 0. \tag{1.10}$$

Erbe and Wang [14], and Lian, Wong and Yeh [18] investigated the problem (1.10) when $\lambda = 1$. Some existence results of positive solutions of problem (1.10) are obtained in [14,15,18] when

$$f(t, x) \in C([0, 1] \times [0, \infty), [0, \infty)).$$

Recall that the boundary value problem for the differential equation

$$x^{\Delta\Delta}(t) + F(t, x(\sigma(t))) = 0, \qquad t \in [a, \sigma(b)], \tag{1.10}$$

is nonsingular if F is continuous in t on $[a, \sigma(b)]$. If $F(t, x)$ is not continuous in t at the end points of $[a, \sigma(b)]$ (including the case that $F(t, x)$ is unbounded on $(a, \sigma(b))$), the above problem is singular.

Accordingly, the problems in [6,13–16,18] are nonsingular. Since singular boundary value problems model a wide spectrum of nonlinear phenomena such as gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysts theory, adiabatic tubular reactor processes (cf., e.g., [19,20]), it is worthwhile to study them deeply. A very recent paper about singular problems is [7].

The aim of this article is to investigate the singular BVP (1.1) under weaker conditions than earlier papers (see (H_1), (H_4), and (H_5) below). The operator approximation method is used in order to deal with the singularity. With the aid of the fixed-point index theorem, theorems on existence of positive solutions to the BVP (1.1) and eigenvalue interval are obtained, which extend some earlier corresponding results in this field (cf. Remark 2.10, below).

We present our main results in Section 2. We first show some lemmas, and then derive existence theorems (Theorems 2.6 and 2.7 below) for positive solutions of the BVP (1.1). Moreover, eigenvalues are determined in Corollaries 2.8 and 2.9 to ensure the existence of positive solution of boundary value problems $(1.1)_\lambda$.

2. MAIN RESULTS

Assume that the set $[a, \sigma(b)]$ is such that both

$$\xi := \min \left\{ \tau \in \mathbb{T} : \tau \geq \frac{\sigma(b) + 3a}{4} \right\}$$

and

$$\omega := \max\left\{\tau \in \mathbb{T} : \tau \le \frac{3\sigma(b)+a}{4}\right\}$$

exist and satisfy

$$\frac{\sigma(b)+3a}{4} \le \xi < \omega \le \frac{3\sigma(b)+a}{4}.$$

We also assume that if $\sigma(\omega) = b$ and $\delta = 0$, then $\sigma(\omega) < \sigma(b)$.

REMARK 2.1. There exists a misprint in the definition of ω on [13, p. 580], and also in lines 9 and 20 on [13, p. 578].

Let $G(t,s)$ be the *Green* function of the following BVP,

$$\begin{aligned} [\rho(t)x^\Delta(t)]^\Delta &= 0, \qquad t \in [a,b],\\ \alpha x(a) - \beta x^\Delta(a) &= 0,\\ \gamma x(\sigma(b)) + \delta x^\Delta(\sigma(b)) &= 0. \end{aligned}$$

From [12] we know that, for any $(t,s) \in [a,\sigma^2(b)] \times [a,b]$,

$$G(t,s) = \begin{cases} \dfrac{1}{d}u(t)v(\sigma(s)), & t \le s,\\ \dfrac{1}{d}u(\sigma(s))v(t), & \sigma(s) \le t, \end{cases}$$

where

$$u(t) = \alpha\int_a^t \frac{1}{\rho(\tau)}\Delta\tau + \frac{\beta}{\rho(a)}, \qquad v(t) = \gamma\int_t^{\sigma(b)} \frac{1}{\rho(\tau)}\Delta\tau + \frac{\delta}{\rho(\sigma(b))}. \tag{2.1}$$

It follows from the monotonicity of functions u and v that, for any $(t,s) \in [a,\sigma^2(b)] \times [a,b]$,

$$\frac{G(t,s)}{G(\sigma(s),s)} = \begin{cases} \dfrac{u(t)}{u(\sigma(s))}, & t \le s,\\ \dfrac{v(t)}{v(\sigma(s))}, & \sigma(s) \le t, \end{cases} \ge \begin{cases} \dfrac{u(t)}{u(\sigma(b))}, & t \le s,\\ \dfrac{v(t)}{v(\sigma(a))}, & \sigma(s) \le t. \end{cases}$$

This implies that

$$G(t,s) \le G(\sigma(s),s), \qquad (t,s) \in [a,\sigma^2(b)] \times [a,b], \tag{2.2}$$

$$G(t,s) \ge kG(\sigma(s),s), \qquad (t,s) \in \left[\frac{\sigma(b)+3a}{4}, \frac{3\sigma(b)+a}{4}\right] \times [a,b], \tag{2.3}$$

where

$$k = \min\left\{\frac{u((\sigma(b)+3a)/4)}{u(\sigma(b))}, \frac{v((3\sigma(b)+a)/4)}{v(\sigma(a))}\right\}.$$

Clearly,

$$l_1 := \min_{s\in[\xi,\omega]} \frac{G(\sigma(\omega),s)}{G(\sigma(s),s)} = \frac{v(\sigma(\omega))}{v(\sigma(\xi))} < 1, \qquad k_1 := \min\{k, l_1\} < 1. \tag{2.4}$$

For the sake of convenience we list the assumptions to be used in this paper as follows.

(H_1) $m(t) : [a,\sigma(b)] \to [0,\infty)$ is continuous in the open interval $(a,\sigma(b))$ which may be singular at $t = a$ and/or $t = \sigma(b)$.

(H_2) $0 < \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s$, $\int_a^{\sigma(b)} G(\sigma(s),s)m(s)\Delta s < \infty$.

(H_3) $f \in C([a,\sigma(b)] \times [0,\infty), [0,\infty))$.

$$(\mathrm{H}_4)\qquad 0 \le f^0 := \limsup_{x\to 0^+} \max_{t\in[a,\sigma(b)]} \frac{f(t,x)}{x} < L,$$

$$l < f_\infty := \liminf_{x\to\infty} \min_{t\in[a,\sigma(b)]} \frac{f(t,x)}{x} \le \infty.$$

$$(\mathrm{H}_5)\qquad 0 \le f^\infty := \limsup_{x\to\infty} \max_{t\in[a,\sigma(b)]} \frac{f(t,x)}{x} < L,$$

$$l < f_0 := \liminf_{x\to 0^+} \min_{t\in[a,\sigma(b)]} \frac{f(t,x)}{x} \le \infty,$$

where

$$L := \frac{1}{\int_a^{\sigma(b)} G(\sigma(s),s)m(s)\Delta s}, \qquad l := \frac{1}{k_1 \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s}.$$

DEFINITION 2.2. *A positive solution to (1.1) is a function $x \in C([a,\sigma^2(b)],\ [0,\infty))$, such that (1.1) holds for $t \in (a,b)$, and also for t at one or both of the two endpoints a and b whenever the function $m(\cdot)$ is continuous there.*

We now define a Banach space

$$E = \{x;\ x : [a,\sigma^2(b)] \to \mathbb{R} \text{is continuous}\}$$

equipped with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{t\in[a,\sigma^2(b)]} |x(t)|.$$

By (2.2) and (H_2), we can define an operator A by

$$Ax(t) = \int_a^{\sigma(b)} G(t,s)m(s)f(s,x(\sigma(s)))\Delta s, \qquad t \in [a,\sigma^2(b)].$$

Define a cone P in the Banach space $(E,\|\cdot\|)$ by

$$P = \left\{x \in E;\ x \ge 0 \text{ on } [a,\sigma^2(b)] \text{ and } \min_{t\in[\xi,\sigma(\omega)]} x(t) \ge k_1\|x\|\right\}.$$

From the fact that $G(t,s)$ is the Green function, we know that the BVP (1.1) has a solution if and only if the operator A has a fixed point.

LEMMA 2.3. *Assume that (H_1)–(H_3) hold. Then, $A : P \to P$ is completely continuous.*

PROOF. Conditions (H_2) and (H_3) imply that $Ax(t) \ge 0$ on $[a,\sigma^2(b)]$, for any $x \in P$. On the other hand, we know by (2.3) that

$$\begin{aligned}\min_{t\in[\xi,\omega]} Ax(t) &= \int_a^{\sigma(b)} \min_{t\in[\xi,\omega]} G(t,s)m(s)f(s,x(\sigma(s)))\Delta s\\ &\ge k\int_a^{\sigma(b)} G(\sigma(s),s)m(s)f(s,x(\sigma(s)))\Delta s.\end{aligned}$$

Thus, by (2.4),

$$\min_{t\in[\xi,\omega]} Ax(t) \ge k\|Ax\| \ge k_1\|Ax\|.$$

Moreover, by (2.4), we have

$$\begin{aligned}Ax(\sigma(\omega)) &= \int_a^{\sigma(b)} G(\sigma(\omega),s)m(s)f(s,x(\sigma(s)))\Delta s\\ &\ge l_1\int_a^{\sigma(b)} G(\sigma(s),s)m(s)f(s,x(\sigma(s)))\Delta s \ge k_1\|Ax\|.\end{aligned}$$

Hence, $A : P \to P$.

Secondly, since $m(t)$ might be singular at $t = a$ and/or $t = \sigma(b)$, we take the following arguments to show that the operator A is completely continuous.

Assume that $x_n, x_* \in P$, $\|x_n - x_*\| \to 0 (n \to \infty)$. Then, there exists $R > 0$, such that $\|x_n\| \le R$, for any $n \ge 1$. Obviously, for all $t \in [a, \sigma^2(b)]$,

$$|Ax_n(t) - Ax_*(t)| \le \int_a^{\sigma(b)} G(\sigma(s), s) m(s) \left| f(s, x_n(\sigma(s))) - f(s, x_*(\sigma(s))) \right| \Delta s.$$

In view of the continuity of f in x, we deduce that

$$|Ax_n(t) - Ax_*(t)| \to 0, \qquad (n \to \infty), \quad t \in \left[a, \sigma^2(b)\right].$$

This means that $A : P \to P$ is continuous.

Take $\{a_n\}_{n=1}^{\infty} \subset (a, b)$, such that $a_n \to a$ as $n \to \infty$ when a is right dense, and let $a_n \equiv a$ for each $n \in N$ when a is right scattered. Moreover, take $\{b_n\}_{n=1}^{\infty} \subset (a, \sigma(b))$, such that $b_n \to \sigma(b)$ as $n \to \infty$ when $\sigma(b)$ is left dense, and let $b_n \equiv \sigma(b)$ for each $n \in N$ when $\sigma(b)$ is left scattered. We define

$$m_n(t) = \begin{cases} \min\{m(t), m(a_n)\}, & a \le t \le a_n, \\ m(t), & a_n \le t \le b_n, \\ \min\{m(t), m(b_n)\}, & b_n \le t \le \sigma(b), \end{cases}$$

and an operator sequence $\{A_n\}$ by

$$A_n x(t) = \int_a^{\sigma(b)} G(t, s) m_n(s) f(s, x(\sigma(s))) \Delta s, \qquad t \in \left[a, \sigma^2(b)\right], \quad n \ge 2$$

Clearly, the operator A_n is compact, for any $n \in N$. Let $R_* > 0$, $B_{R_*} := \{x \in P : \|x\| \le R_*\}$. We shall prove that A_n approach A uniformly on B_{R_*}.

In fact, we infer, by (2.1), that the function v is decreasing and the function u is increasing. Thus, for any $x \in B_{R_*}$, $a < t \le a_n$, we have

$$\begin{aligned} |A_n x(t) - Ax(t)| &= \left| \frac{1}{d} v(t) \int_a^t u(\sigma(s)) \left[m(s) - m_n(s)\right] f(s, x(\sigma(s))) \Delta s \right. \\ &\quad + \frac{1}{d} u(t) \int_t^{a_n} v(\sigma(s)) [m(s) - m_n(s)] f(s, x(\sigma(s))) \Delta s \\ &\quad \left. + \frac{1}{d} u(t) \int_{b_n}^{\sigma(b)} v(\sigma(s)) [m(s) - m_n(s)] f(s, x(\sigma(s))) \Delta s \right| \\ &\le \frac{1}{d} v(t) \int_a^t u(\sigma(s)) |m(s) - m_n(s)| f(s, x(\sigma(s))) \Delta s \\ &\quad + \frac{1}{d} v(t) \int_t^{a_n} u(\sigma(s)) |m(s) - m_n(s)| f(s, x(\sigma(s))) \Delta s \\ &\quad + \frac{1}{d} u(t) \int_{b_n}^{\sigma(b)} v(\sigma(s)) |m(s) - m_n(s)| f(s, x(\sigma(s))) \Delta s \\ &\le M^* \left[\int_a^{a_n} u(\sigma(s)) |m(s) - m_n(s)| \Delta s \right. \\ &\quad \left. + \int_{b_n}^{\sigma(b)} v(\sigma(s)) [m(s) - m_n(s)] \Delta s \right], \end{aligned}$$

where

$$M^* := \frac{1}{d} \max\{u(\sigma^2(b)), v(a)\}.$$

Similarly, we have

$$|A_n x(t) - Ax(t)| \le M^* \left[\int_a^{a_n} u(\sigma(s))|m(s) - m_n(s)|\Delta s + \int_{b_n}^{\sigma(b)} v(\sigma(s))[m(s) - m_n(s)]\Delta s \right], \tag{2.5}$$

when $a_n \le t \le b_n$ or $b_n \le t \le \sigma^2(b)$. Thus, (2.5) holds, for any $t \in [a, \sigma^2(b)]$. Assumption (H_2), together with the fact that $0 \le m_n(s) \le m(s)$, implies that the right-hand side can be sufficiently small for n big enough. Therefore, the sequence $\{A_n\}$ of compact operators converges to A uniformly on B_{R_*}. So, the operator A is compact.

Consequently, A is completely continuous. ∎

The following two fixed-point index theorems are not of norm-type, which are important in setting up our results.

LEMMA 2.4. *(See [21].) Let P be a cone in a Banach space X, $\Omega \subset X$ a bounded set, and $A : \bar{\Omega} \cap P \to P$ a completely continuous operator. If $Ax \ne \lambda x$, for any $x \in \partial\Omega \cap P, \lambda \ge 1$, then the fixed-point index $i(A, \Omega \cap P, P) = 1$.*

LEMMA 2.5. *(See [22].) Let P be a cone in a Banach space X, $\Omega \subset X$ a bounded set, and $A : \bar{\Omega} \cap P \to P$ a completely continuous operator. If there exists an operator $B : \partial\Omega \cap P \to P$, such that*

(i) $\inf_{x \in \partial\Omega \cap P} \|Bx\| > 0$;
(ii) $x - Ax \ne \lambda Bx$, *for any* $x \in \partial\Omega \cap P$, $\lambda \ge 0$,

then the fixed-point index $i(A, \Omega \cap P, P) = 0$.

Now, we are in a position to present and prove our main results.

THEOREM 2.6. *Assume that (H_1)–(H_4) hold. Then, the BVP (1.1) has at least one positive solution.*

PROOF. By (H_1)–(H_3) we know that Lemma 2.3 holds.

By the first inequality of (H_4), there exist $r_1 > 0$ and $\varepsilon_1 > 0$, such that

$$\max_{t \in [a, \sigma(b)]} f(t, x) \le (L - \varepsilon_1)x, \quad 0 \le x \le r_1.$$

Hence,

$$f(t, x) \le (L - \varepsilon_1)x, \qquad 0 \le x \le r_1, \quad a \le t \le \sigma(b). \tag{2.6}$$

Let $\Omega_1 := \{x \in E : \|x\| < r_1\}$. We know that, for any $x \in \partial\Omega_1 \cap P$,

$$\begin{aligned}
\|Ax\| &= \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s)m(s)f(s, x(\sigma(s)))\Delta s \\
&\le (L - \varepsilon_1)r_1 \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s)m(s)\Delta s \qquad \text{(using (2.6))} \\
&\le (L - \varepsilon_1)r_1 \int_a^{\sigma(b)} G(\sigma(s), s)m(s)\Delta s \\
&= r_1 L \int_a^{\sigma(b)} G(\sigma(s), s)m(s)\Delta s - r_1\varepsilon_1 \int_a^{\sigma(b)} G(\sigma(s), s)m(s)\Delta s \\
&< r_1.
\end{aligned} \tag{2.7}$$

We claim that

$$Ax \ne \lambda x, \qquad \forall x \in \partial\Omega_1 \cap P, \quad \lambda \ge 1. \tag{2.8}$$

In fact, if (2.8) is not true, then there exist $x_1 \in \partial\Omega_1 \cap P$ and $\lambda_1 \geq 1$, such that $Ax_1 = \lambda_1 x_1$. Thus,

$$\|Ax_1\| = \lambda_1 \|x_1\| \geq \|x_1\| = r_1.$$

Obviously, this is in contradiction with (2.7). Hence, (2.8) holds. Using (2.8) and Lemma 2.4, we have

$$i(A, \Omega_1 \cap P, P) = 1. \tag{2.9}$$

By the second inequality of (H_4), there exist $\eta > k_1 r_1 > 0$ and $\varepsilon_2 > 0$, such that

$$\min_{t\in[a,\sigma(b)]} f(t,x) \geq (l+\varepsilon_2)\,x, \qquad x \geq \eta.$$

Therefore,

$$\min_{t\in[\xi,\omega]} f(t,x) \geq (l+\varepsilon_2)x, \qquad x \geq \eta,$$

that is,

$$f(t,x) \geq (l+\varepsilon_2)\,x, \qquad x \geq \eta, \quad t \in [\xi,\omega]. \tag{2.10}$$

Write

$$r_2 = \frac{\eta}{k_1} > r_1, \qquad \Omega_2 := \{x \in E : \|x\| < r_2\},$$

and define an operator B by

$$Bx(t) \equiv 1, \qquad x \in E. \tag{2.11}$$

It's easy to see that $B : \partial\Omega_2 \cap P \to P$ is completely continuous. Moreover, we know that $\inf_{x\in\partial\Omega_2\cap P} \|Bx\| > 0$. So, Condition (i) of Lemma 2.5 holds. Next we verify Condition (ii) of Lemma 2.5

$$x - Ax \neq \lambda Bx, \qquad \forall\, x \in \partial\Omega \cap P, \quad \forall\, \lambda \geq 0. \tag{2.12}$$

Suppose that (2.12) is false, then there exist $x_2 \in \partial\Omega_2 \cap P$ and $\lambda_2 \geq 0$, such that

$$x_2 - Ax_2 = \lambda_2 Bx_2.$$

Clearly,

$$\min\{x_2(\sigma(s)) : s \in [\xi,\omega]\} \geq \min\{x_2(s) : s \in [\xi,\sigma(\omega)]\} \geq k_1\|x_2\| = \eta.$$

Therefore, we have by (2.10) that

$$f(t, x_2(\sigma(s))) \geq (l+\varepsilon_2)x_2(\sigma(s)), \qquad (s,t) \in [\xi,\omega] \times [\xi,\omega]. \tag{2.13}$$

Set

$$C := \min\{x_2(t) : t \in [\xi,\sigma(\omega)]\}. \tag{2.14}$$

Then, $C > 0$. By (2.3), (2.11), (2.13), and (2.14), we obtain, for any $t \in [\xi,\sigma(\omega)]$,

$$\begin{aligned}
x_2(t) &= \int_a^{\sigma(b)} G(t,s)m(s)f(s,x_2(\sigma(s)))\Delta s + \lambda_2 Bx_2(t) \\
&\geq \int_\xi^\omega G(t,s)m(s)f(s,x_2(\sigma(s)))\Delta s + \lambda_2 \\
&\geq \int_\xi^\omega kG(\sigma(s),s)m(s)f(s,x_2(\sigma(s)))\Delta s + \lambda_2 \qquad (\text{ using } (2.3)) \\
&\geq k_1 \int_\xi^\omega G(\sigma(s),s)m(s)(l+\varepsilon_2)x_2(\sigma(s))\Delta s \qquad (\text{using } (2.13)) \\
&\geq k_1(l+\varepsilon_2) \min_{s\in[\xi,\omega]} x_2(\sigma(s)) \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s \\
&\geq k_1(l+\varepsilon_2) \min_{s\in[\xi,\sigma(\omega)]} x_2(s) \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s \\
&= Clk_1 \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s + C\varepsilon_2 k_1 \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s. \\
&= C + C\varepsilon_2 k_1 \int_\xi^\omega G(\sigma(s),s)m(s)\Delta s.
\end{aligned} \tag{2.15}$$

(2.15) and the first inequality of (H_2) imply

$$x_2(t) > C, \qquad \forall t \in [\xi, \sigma(\omega)]. \tag{2.16}$$

Obviously, (2.16) contradicts (2.14). This means that (2.12) holds.

Now applying Lemma 2.5, we get

$$i(A, \Omega_2 \cap P, P) = 0. \tag{2.17}$$

Combining (2.9),(2.17) and the fact that $\bar{\Omega}_1 \subset \Omega_2$, we have

$$i(A, (\Omega_2 \setminus \bar{\Omega}_1) \cap P, P) = i(A, \Omega_2 \cap P, P) - i(A, \Omega_1 \cap P, P) = 0 - 1 = -1.$$

By virtue of [22, Theorem 2.3.2], we know that the operator A has a fixed-point $x^* \in (\Omega_2 \setminus \bar{\Omega}_1) \cap P$, such that $0 < r_1 \le \|x^*\| \le r_2$. It follows that x^* is a positive solution of the BVP(1.1). ∎

THEOREM 2.7. *Assume that (H_1)–(H_3), and (H_5) hold. Then, the BVP (1.1) has at least one positive solution.*

PROOF. Assumptions (H_1)–(H_3) imply that Lemma 2.3 holds.

By the first inequality of (H_5), there exist $\mu > 0$ and $\varepsilon_3 > 0$, such that

$$\max_{t \in [a, \sigma(b)]} f(t, x) \le (L - \varepsilon_3)x, \qquad x \ge \mu.$$

Thus,

$$f(t, x) \le (L - \varepsilon_3)x, \qquad x \ge \mu, \quad a \le t \le \sigma(b).$$

Set

$$\zeta := \max_{(t,x) \in [a, \sigma(b)] \times [0, \mu]} f(t, x).$$

Then,

$$f(t, x) \le \zeta + (L - \varepsilon_3)x, \qquad (t, x) \in [a, \sigma(b)] \times [0, \infty). \tag{2.18}$$

Let $r_3 > \zeta/\varepsilon_3$ and $\Omega_3 := \{x \in E : \|x\| < r_3\}$. Then, we get, by (2.18), for any $x \in \partial\Omega_3 \cap P$,

$$\begin{aligned} \|Ax\| &= \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s) m(s) f(s, x(\sigma(s))) \Delta s \\ &\le [\zeta + (L - \varepsilon_3) r_3] \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} G(t, s) m(s) \Delta s \\ &\le r_3 L \int_a^{\sigma(b)} G(\sigma(s), s) m(s) \Delta s - (r_3 \varepsilon_3 - \zeta) \int_a^{\sigma(b)} G(\sigma(s), s) m(s) \Delta s \\ &< r_3. \end{aligned} \tag{2.19}$$

Therefore,

$$Ax \ne \lambda x, \qquad \forall x \in \partial\Omega_3 \cap P, \quad \lambda \ge 1. \tag{2.20}$$

In fact, if (2.20) is not true, then there exist $x_1 \in \partial\Omega_3 \cap P$ and $\lambda_1 \ge 1$, such that $Ax_1 = \lambda_1 x_1$. Thus,

$$\|Ax_1\| = \lambda_1 \|x_1\| \ge \|x_1\| = r_3.$$

This is in contradiction with (2.19). So (2.20) holds. It follows from (2.20) and Lemma 2.4 that

$$i(A, \Omega_3 \cap P, P) = 1. \tag{2.21}$$

By the second inequality of (H_5), there exist $0 < \eta^* < (k_1+1)r_3$ and $\varepsilon_4 > 0$, such that

$$\min_{t\in[a,\sigma(b)]} f(t,x) \geq (l+\varepsilon_4)x, \qquad 0 \leq x \leq \eta^*.$$

Then,

$$f(t,x) \geq (l+\varepsilon_4)x, \qquad 0 \leq x \leq \eta^*, \quad t \in [a,\sigma(b)]. \tag{2.22}$$

Let

$$r_4 = \frac{\eta^*}{k_1+1} < r_3, \qquad \Omega_4 := \{x \in E : \|x\| < r_4\}.$$

Define an operator B as in (2.11). Then, $B : \partial\Omega_4 \cap P \to P$ is completely continuous and $\inf_{x\in\partial\Omega_4\cap P} \|Bx\| > 0$. Hence, the Condition (i) of Lemma 2.5 holds. We claim that Condition (ii) of Lemma 2.5

$$x - Ax \neq \lambda Bx, \qquad \forall\, x \in \partial\Omega \cap P, \quad \forall\, \lambda \geq 0 \tag{2.23}$$

holds too. If not, then there exist $x_4 \in \partial\Omega_4 \cap P$ and $\lambda_4 \geq 0$, such that $x_4 - Ax_4 = \lambda_4 Bx_4$. Set

$$C^* := \min\{x_4(t) : t \in [\xi,\sigma(\omega)]\}. \tag{2.24}$$

Then, $C^* \geq k_1\|x_4\| = k_1 r_4 > 0$. We have, by a similar argument used to prove (2.15), for any $t \in [\xi,\sigma(\omega)]$,

$$\begin{aligned}
x_4(t) &= \int_a^{\sigma(b)} G(t,s)m(s)f(s,x_4(\sigma(s)))\Delta s + \lambda_4 Bx_4(t) \\
&\geq \int_a^{\sigma(b)} G(t,s)m(s)(l+\varepsilon_4)x_4(\sigma(s))\Delta s + \lambda_4 \qquad \text{(using (2.22))} \\
&\geq \int_\xi^{\omega} G(t,s)m(s)(l+\varepsilon_4)x_4(\sigma(s))\Delta s \\
&\geq \int_\xi^{\omega} kG(\sigma(s),s)m(s)(l+\varepsilon_4)x_4(\sigma(s))\Delta s \\
&\geq k_1(l+\varepsilon_4)\min_{s\in[\xi,\omega]} x_4(\sigma(s)) \int_\xi^{\omega} G(\sigma(s),s)m(s)\Delta s \\
&\geq k_1(l+\varepsilon_4)\min_{s\in[\xi,\sigma(\omega)]} x_4(s) \int_\xi^{\omega} G(\sigma(s),s)m(s)\Delta s \\
&= C^* k_1(l+\varepsilon_4)\int_\xi^{\omega} G(\sigma(s),s)m(s)\Delta s \\
&= C^* + C^*\varepsilon_4 k_1 \int_\xi^{\omega} G(\sigma(s),s)m(s)\Delta s.
\end{aligned}$$

Noting that $\int_\xi^\omega G(\sigma(s),s)m(s)\Delta s > 0$, we get

$$x_4(t) > C^*, \qquad \forall\, t \in [\xi,\sigma(\omega)]. \tag{2.25}$$

Clearly, (2.25) contradicts (2.24). So (2.23) holds.

Then, the conditions of Lemma 2.5 are satisfied. This gives

$$i(A,\Omega_4 \cap P,P) = 0. \tag{2.26}$$

From $\bar{\Omega}_4 \subset \Omega_3$, (2.21) and (2.26), it follows that

$$i(A,(\Omega_3 \setminus \bar{\Omega}_4)\cap P,P) = i(A,\Omega_3\cap P,P) - i(A,\Omega_4 \cap P,P) = 1 - 0 = 1.$$

In view of [23, Theorem 2.3.2], we see that the operator A has a fixed-point $x^* \in P$, such that $0 < r_4 \leq \|x^*\| \leq r_3$. This x^* is a positive solution of the BVP (1.1). ∎

The following corollaries are direct consequences of Theorem 2.6 and Theorem 2.7.

COROLLARY 2.8. *Assume that (H$_1$)–(H$_4$) are satisfied. Then, for each λ satisfying*

$$\lambda \in \left(\frac{l}{f_\infty}, \frac{L}{f^0}\right), \tag{2.27}$$

there exists at least one positive solution of the BVP (1.1)$_\lambda$.

PROOF. For any λ satisfying (2.27), we have

$$0 \le \frac{lf^0}{f_\infty} \le \lambda f^0 < L, \qquad l < \lambda f_\infty \le \frac{Lf_\infty}{f^0} \le \infty. \tag{2.28}$$

Thus, an application of Theorem 2.6 to (2.28) yields the desired conclusion. ∎

COROLLARY 2.9. *Assume that (H$_1$)–(H$_3$), and (H$_5$) are satisfied. Then, for each λ satisfying*

$$\lambda \in \left(\frac{l}{f_0}, \frac{L}{f^\infty}\right),$$

there exists at least one positive solution of the BVP (1.1)$_\lambda$.

PROOF. This proof is similar to that of Corollary 2.8.

REMARK 2.10.

(1) It seems to be difficult to utilize the norm-type expansion and compression theorems to prove our Theorem 2.6 and Theorem 2.7.
(2) The function $m(t) : (a, \sigma(b)) \to [0, \infty)$ in Assumption (H$_1$) might vanish on some subintervals of $(a, \sigma(b))$.
(3) (H$_4$) and (H$_5$) in this paper relax some superlinear or sublinear conditions in earlier results in this field.
(4) The current work is a generalization of [9].
(5) Let

$$a = 0, \qquad b = 1, \qquad \rho(t) \equiv 1, \qquad m(t) \equiv 1.$$

Then, problem $(1.1)_\lambda$ reduces to the BVP in [16], and Corollary 2.8 and Corollary 2.9 go back to Theorem 1.8 and Theorem 1.9, respectively. Moreover, in this case, the integral $\int_\xi^\omega G(\tau, s)\Delta s$ in [16] is less than $\int_\xi^\omega G(\sigma(s), s)m(s)\Delta s$ of this paper, since $G(\tau, s) \le G(\sigma(s), s)$. Thus, the interval I_f in Theorem 1.8 is a subset of interval $(l/f_\infty, L/f^0)$ in this paper (see Corollary 2.8). The same comment applies to Corollary 2.9 and Theorem 1.9. This indicates that the intervals of λ to ensure the existence of positive solutions in this paper could be bigger.

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